ON THE SPLITTING PROBLEM FOR MANIFOLD PAIRS WITH BOUNDARIES

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Abstract. The problem of splitting a homotopy equivalence along a submanifold is closely related to the surgery exact sequence and to the problem of surgery of manifold pairs. In classical surgery theory there exist two approaches to surgery in the category of manifolds with boundaries. In the *rel* ∂ case the surgery on a manifold pair is considered with the given fixed manifold structure on the boundary. In the relative case the surgery on the manifold with boundary is considered without fixing maps on the boundary. Consider a normal map to a manifold pair $(Y, \partial Y) \subset (X, \partial X)$ with boundary which is a simple homotopy equivalence on the boundary ∂X. This map defines a mixed structure on the manifold with the boundary in the sense of Wall. We introduce and study groups of obstructions to splitting of such mixed structures along submanifold with boundary $(Y, \partial Y)$. We describe relations of these groups to classical surgery and splitting obstruction groups. We also consider several geometric examples.

1. Introduction.

Let $(X^n, \partial X)$ be a compact topological n–manifold with boundary. The set $\mathcal{S}^{CAT}(X, \partial X)$ of CAT-manifold structures ($CAT = TOP, PL, DIFF$) on $(X, \partial X)$ consists of the classes of concordance of simple homotopy equivalences of pairs $f\colon (M, \partial M) \to (X, \partial X)$, where $(M, \partial M)$ is a compact CAT-manifold pair of dimension n with boundary (see [7], [8], and [11]). If ∂X already has a CAT-manifold structure then the set of manifold structures on X which are fixed on the boundary is denoted by $\mathcal{S}^{CAT}_\partial(X, \partial X)$.

Let $\mathcal{T}^{CAT}(X, \partial X)$ be the set of classes of normal bordisms of normal maps to the pair $(X, \partial X)$ and $\mathcal{T}^{CAT}_\partial(X, \partial X)$ the set of *rel* ∂ classes of normal bordisms of normal maps (see [7], [8] and [11]).

Let $Y \subset X$ be a submanifold of a closed manifold X^n of codimension q. Given a normal map $(f, b) : M^n \to X^n$, there is a problem of finding a simple homotopy equivalence $g : M_1 \to X$ in the class of normal bordism $[(f, b)]$, which is transversal to Y and such that $N = g^{-1}(Y)$ is a submanifold of M_1 and the restrictions

$$g|_N : N \to Y,\ g|_{M_1\setminus N} : M_1 \setminus N \to X \setminus Y \tag{1.1}$$

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are simple homotopy equivalences. The obstruction group for doing such surgery is denoted by $LP_{n-q}(F)$ (see [11] and [8]), where

$$F = \begin{pmatrix} \pi_1(S(\xi)) & \longrightarrow & \pi_1(X \setminus Y) \\ \downarrow & & \downarrow \\ \pi_1(D(\xi)) & \longrightarrow & \pi_1(X) \end{pmatrix} \tag{1.2}$$

is a pushout square of fundamental groups with orientations and $S(\xi)$ is the boundary of a tubular neighborhood $D(\xi)$ of Y in X.

If $f : M \to X$ is a simple homotopy equivalence then the obstruction to finding a map in the homotopy class of the map f with properties (1.1), which is transversal to Y, lies in the splitting obstruction group $LS_{n-q}(F)$ (see [11, §11] and [8, §7.2]).

Let $(Y, \partial Y) \subset (X, \partial X)$ be a codimension q manifold pair with boundary. In the *rel* ∂ case the set $\mathcal{T}^{CAT}_\partial(X, \partial X)$ of tangent structures consists of classes of concordance rel boundary of normal maps

$$(f, \partial f) : (M, \partial M) \to (X, \partial X)$$

with a fixed CAT-isomorphism

$$\partial f : \partial M \to \partial X$$

which is already split on the boundary. We have a map

$$\mathcal{T}^{CAT}_\partial(X, \partial X) \to LP_{n-q}(F) \tag{1.3}$$

which is given by mapping the obstruction to surgery to the normal map of manifold pairs rel boundary.

In a similar way we have a map

$$\mathcal{S}^{CAT}_\partial(X, \partial X) \to LS_{n-q}(F) \tag{1.4}$$

to the splitting obstruction group.

It follows from [11, §11, page 136] (see also [12]) that for the relative case we have maps

$$\mathcal{T}^{CAT}(X, \partial X) \to LP_{n-q}(F_\partial \to F) \tag{1.5}$$

similarly to (1.3) and (1.4) and

$$\mathcal{S}^{CAT}(X, \partial X) \to LS_{n-q}(F_\partial \to F) \tag{1.6}$$

to the relative obstruction groups where F_∂ is a pushout square for a splitting problem of the pair $\partial Y \subset \partial X$.

In accordance with Wall [11, §10, p. 116] (see also [3]), it is possible to introduce a mixed type of structures on a manifold with boundary $(X, \partial X)$. Consider a normal map

$$(f, \partial f) : (M, \partial M) \to (X, \partial X)$$

for which the map

$$\partial f : \partial M \to \partial X$$

is a simple homotopy equivalence. Two such maps are concordant if they are normally bordant by a bordism for which a restriction to bordism between the boundaries is an equivalence in $S^{CAT}(\partial X)$. Denote the set of concordance classes by $\mathcal{TS}^{CAT}(X, \partial X)$. The elements of $\mathcal{TS}^{CAT}(X, \partial X)$ are called mixed structures on $(X, \partial X)$.

In the present paper we shall work in TOP-category and simple surgery obstruction groups (see [7] and [8]). We think of all surgery and splitting obstruction groups as decorated by an "s" although we do not write it.

We introduce groups $LPS_*(F_\partial \to F)$ and define a map

$$\psi : \mathcal{TS}(X, \partial X) \to LPS_*(F_\partial \to F)$$

which gives an obstruction to finding a map in the class of concordance which is split along the submanifold pair $(Y, \partial Y) \subset (X, \partial X)$.

We study properties of the introduced groups and their relations to surgery and splitting obstruction groups. The main results are given by braids of exact sequences. Then we consider several geometric examples in which we compute the introduced groups and natural maps.

In Section 2 we give explicit definitions of several structure sets and recall the necessary technical results about the algebraic surgery exact sequences of Ranicki and the surgery L-spectrum.

In Section 3 we recall main properties of splitting obstruction groups and introduce LPS_*-groups. These groups are realized as homotopy groups of a spectrum. We describe algebraic properties of these groups and relations of these groups to surgery and splitting obstruction groups and to surgery exact sequence.

In Section 4 we consider geometric examples in which we compute LPS_*-groups and natural maps between introduced groups and classical obstruction groups which arise naturally in the considered problem.

2. Structure sets and surgery exact sequence.

For definitions of structure sets we shall follow Ranicki [8]. Let X^n be a closed topological manifold. A t-triangulation of X is a topological normal map (see [8, §1.2])

$$(f, b) : M \to X,$$

where M is a closed n-dimensional topological manifold. Two t-triangulations

$$(f_i, b_i) : M_i \to X, \; i = 0, 1$$

are concordant [8, §7.1] if there exists a topological normal map of triads

$$((g, c); (f_0, b_0), (f_1, b_1) : (W; M_0, M_1) \to (X \times I; X \times \{0\}, X \times \{1\})$$

where $I = [0, 1]$ and W is a compact $(n+1)$-dimensional manifold with boundary $\partial W = M_0 \cup M_1$. The set of concordance classes of t-triangulations of X is denoted by $\mathcal{T}^{TOP}(X)$. Note that we shall consider the case of a manifold X and hence the set $\mathcal{T}^{TOP}(X)$ will be nonempty.

An s-triangulation of a closed topological manifold X^n is a simple homotopy equivalence $f : M \to X$, where M is a closed topological n-dimensional manifold.

Two s-triangulations

$$(f_i, b_i) : M_i \to X,\ i = 0, 1$$

are concordant [8, §7.1] if there exists a simple homotopy equivalence of triads

$$(g; f_0, f_1) : (W; M_0, M_1) \to (X \times I; X \times \{0\}, X \times \{1\})$$

where W is a compact $(n+1)$-dimensional manifold with the boundary $\partial W = M_0 \cup M_1$. The set of concordance classes of s-triangulations of X is denoted by $\mathcal{S}^{TOP}(X)$. This set is called the *topological manifold structure set*. As before, the set $\mathcal{S}^{TOP}(X)$ will be nonempty. These sets fit in the surgery exact sequence (see [8, §7] and [11])

$$\cdots L_{n+1}(\pi_1(X)) \to \mathcal{S}^{TOP}(X) \to \mathcal{T}^{TOP}(X) \to L_n(\pi_1(X)) \tag{2.1}$$

where $L_*(\pi_1(X))$ are surgery obstruction groups.

Now consider the case of a compact n-dimensional manifold X with the boundary ∂X. First, we consider the case of structures which are fixed on the boundary. This is the *rel* ∂ case. A t_∂-triangulation of $(X, \partial X)$ is a topological normal map of pairs (see [8, §7.1])

$$((f, b), (\partial f, \partial b)) : (M, \partial M) \to (X, \partial X)$$

with a homeomorphism $\partial f : \partial M \to \partial X$. Two t_∂-triangulations

$$((f_i, b_i), (\partial f_i, \partial b_i)) : (M_i, \partial M_i) \to (X, \partial X),\ i = 0, 1$$

are concordant if there exists a topological normal map

$$\begin{array}{lcl} ((h,d);(g,c),(f_0,b_0),(f_1,b_1)) : & & \\ \qquad (W; V, M_0, M_1) & \longrightarrow & (X \times I; \partial X \times I, X \times \{0\}, X \times \{1\}) \end{array}$$

with

$$V = \partial M_0 \times I,\ \partial V = \partial M_0 \cup \partial M_1$$

and

$$(g, c) = \partial f_0 \times I : V \to \partial X \times I.$$

The set of concordance classes is denoted by $\mathcal{T}_\partial^{CAT}(X, \partial X)$ (see [11, §10] and [8, §7.1]).

An s_∂-triangulation of $(X, \partial X)$ is a simple homotopy equivalence of pairs (see [8, §7.1])

$$(f, \partial f) : (M, \partial M) \to (X, \partial X)$$

with a homeomorphism $\partial f : \partial M \to \partial X$. Two s_∂-triangulations

$$(f_i, \partial f_i) : (M_i, \partial M_i) \to (X, \partial X),\ i = 0, 1$$

are concordant if there exists a simple homotopy equivalence of 4-ads

$$(h; g, f_0, f_1) : (W; V, M_0, M_1) \to (X \times I; \partial X \times I, X \times \{0\}, X \times \{1\})$$

with

$$V = \partial M_0 \times I,\ \partial V = \partial M_0 \cup \partial M_1$$

and

$$g = \partial f_0 \times I : V \to \partial X \times I.$$

The set of concordance classes is denoted by $\mathcal{S}^{CAT}_{\partial}(X, \partial X)$ (see [11, §10] and [8, §7.1]).

These sets fit in the surgery exact sequence (see [11, §10] and [8, §7])

$$\cdots \to L_{n+1}(\pi_1(X)) \to \mathcal{S}^{TOP}_{\partial}(X, \partial X) \to \mathcal{T}^{TOP}_{\partial}(X, \partial X) \to L_n(\pi_1(X)). \tag{2.2}$$

Now consider the relative case of structures on a manifold with boundary. A t-triangulation of $(X, \partial X)$ is a topological normal map of pairs (see [8, §7.1])

$$((f, b), (\partial f, \partial b)) : (M, \partial M) \to (X, \partial X).$$

Two t-triangulations

$$((f_i, b_i), (\partial f_i, \partial b_i)) : (M_i, \partial M_i) \to (X, \partial X),\ i = 0, 1$$

are concordant if there exists a topological normal map of 4-ads

$$((h, d); (g, c), (f_0, b_0), (f_1, b_1)) : (W; V, M_0, M_1) \to (X \times I; \partial X \times I, X \times \{0\}, X \times \{1\})$$

with

$$\partial V = \partial M_0 \cup \partial M_1.$$

The set of concordance classes is denoted by $\mathcal{T}^{CAT}(X, \partial X)$ (see [11, §10] and [8, §7.1]).

An s-triangulation of $(X, \partial X)$ is a simple homotopy equivalence of pairs (see [8, §7.1])

$$(f, \partial f) : (M, \partial M) \to (X, \partial X).$$

Two s-triangulations

$$(f_i, \partial f_i) : (M_i, \partial M_i) \to (X, \partial X),\ i = 0, 1$$

are concordant if there exists a simple homotopy equivalence of 4-ads

$$((h; g, f_0, f_1) : (W; V, M_0, M_1) \to (X \times I; \partial X \times I, X \times \{0\}, X \times \{1\})$$

with

$$\partial V = \partial M_0 \cup \partial M_1.$$

The set of concordance classes is denoted by $\mathcal{S}^{CAT}(X, \partial X)$ (see [11, §10] and [8, §7.1]).

These sets fit in the surgery exact sequence (see [11, §10] and [8, §7])

$$\cdots \to \mathcal{S}^{TOP}(X, \partial X) \to \mathcal{T}^{TOP}(X, \partial X) \to L_n\left(\pi_1(\partial X) \to \pi_1(X)\right). \tag{2.3}$$

Now we define mixed structures on a manifold with boundary (see [11, page 116] and [3]). A t_s-triangulation of $(X, \partial X)$ is a topological normal map of pairs

$$((f, b), (\partial f, \partial b)) : (M, \partial M) \to (X, \partial X)$$

such that $\partial f : \partial M \to \partial X$ is an s-triangulation. Two t_s-triangulations

$$((f_i, b_i), (\partial f_i, \partial b_i)) : (M_i, \partial M_i) \to (X, \partial X),\ i = 0, 1$$

are concordant if there exists a topological normal map

$$\begin{array}{lcl} ((h,d);(g,c),(f_0,b_0),(f_1,b_1)) : & & \\ \qquad (W;V,M_0,M_1) & \longrightarrow & (X\times I;\partial X\times I, X\times\{0\}, X\times\{1\}) \end{array}$$

with

$$\partial V = \partial M_0 \cup \partial M_1$$

and $g : V \to \partial X \times I$ is a concordance of s-triangulations ∂f_0 and ∂f_1. The set of concordance classes is denoted by $\mathcal{TS}^{TOP}(X, \partial X)$ (see [11, page 116]).

It follows from definitions (see also [3]) that the following natural forgetful maps

$$\mathcal{TS}^{TOP}(X, \partial X) \to \mathcal{T}^{TOP}(X, \partial X),$$

$$\mathcal{S}^{TOP}(X, \partial X) \to \mathcal{TS}^{TOP}(X, \partial X), \tag{2.4}$$

$$\mathcal{TS}^{TOP}(X, \partial X) \to \mathcal{S}^{TOP}(\partial X)$$

are well–defined.

The maps in (2.4) fit in the following exact sequences (see [11, page 116], [8, §7], and [3])

$$\cdots \to L_n(\pi_1(\partial X)) \to \mathcal{TS}^{TOP}(X, \partial X) \to \mathcal{T}^{TOP}(X, \partial X) \to L_{n-1}(\pi_1(\partial X)), \tag{2.5}$$

$$\cdots \to L_{n+1}(\pi_1(X)) \to \mathcal{S}^{TOP}(X, \partial X) \to \mathcal{TS}^{TOP}(X, \partial X) \to L_n(\pi_1(X)), \tag{2.6}$$

and

$$\cdots \to \mathcal{T}_{\partial}^{TOP}(X, \partial X) \to \mathcal{TS}^{TOP}(X, \partial X) \to \mathcal{S}^{TOP}(\partial X). \tag{2.7}$$

We now recall the necessary results concerning the application of homotopy category of spectra to surgery theory (see [1], [2], [4], [5], [6], and [7]). A spectrum $\mathbb{E}$ is given by a collection of CW-complexes $\{(E_n, *)\}$, $n \in \mathbb{Z}$, together with cellular maps $\{\epsilon_n : SE_n \to E_{n+1}\}$, where SE_n is the suspension of the space E_n (see [14]). A spectrum $\mathbb{E}$ is an Ω-spectrum if the adjoint maps $\epsilon_n' : E_n \to \Omega E_{n+1},\ n \in \mathbb{Z}$ are homotopy equivalences.

In the category of spectra the suspension functor Σ and iterated functors Σ^k, $k \in \mathbb{Z}$ are well-defined (see [10]). For every spectrum $\mathbb{E}$ we have an isomorphism of homotopy groups $\pi_n(\mathbb{E}) = \pi_{n+k}(\Sigma^k\mathbb{E})$. Recall that in the homotopy category of spectra the concepts of pull-back and push-out squares are equivalent.

In accordance with [7], [8], and [11] the surgery obstruction groups $L_n(\pi)$ and such natural maps as induced by inclusion and transfer are realized on the spectrum level. That is, for every group π with a homomorphism of orientation $\omega : \pi \to \{\pm 1\}$ there exists an Ω-spectrum $\mathbb{L}(\pi, \omega)$ with homotopy groups

$$\pi_n(\mathbb{L}(\pi, \omega)) = L_n(\pi, \omega).$$

In what follows we shall not include homomorphism of orientation in our notation and will assume that all groups are equipped with such a homomorphism and all homomorphisms of groups preserve orientation. Any homomorphism of groups $f : \pi \to G$ induces a cofibration of spectra

$$\mathbb{L}(\pi) \to \mathbb{L}(G) \to \mathbb{L}(f) \tag{2.8}$$

where $\mathbb{L}(f)$ is the spectrum for the relative L-groups

$$L_n(f) = L_n(\pi \to G) = \pi_n(\mathbb{L}(f)).$$

We have a similar situation for the transfer map (see for example, [11] and [12]).

Let X be a topological space. An algebraic surgery exact sequence of Ranicki (see [7] and [8])

$$\cdots \to L_{n+1}(\pi_1(X)) \to \mathcal{S}_{n+1}(X) \to H_n(X, \mathbf{L}_\bullet) \to L_n(\pi_1(X)) \to \cdots \tag{2.9}$$

is defined. Here $\mathbf{L}_\bullet$ is the 1-connected cover of the surgery Ω-spectrum $\mathbb{L}(1)$ with $\{\mathbf{L}_\bullet\}_0 \simeq G/TOP$. The algebraic surgery exact sequence (2.9) is the homotopy long exact sequence of the cofibration

$$X_+ \wedge \mathbf{L}_\bullet \to \mathbb{L}(\pi_1(X)). \tag{2.10}$$

By definition, we have $\mathcal{S}_i(X) = \pi_i(\mathbb{S}(X))$ for the homotopy cofiber $\mathbb{S}(X)$ of the map in (2.10). For a closed n-dimensional topological manifold X we have

$$\pi_{n+1}(\mathbb{S}(X)) = \mathcal{S}_{n+1}(X) \cong \mathcal{S}^{TOP}(X),\ \mathcal{T}(X) \cong H_n(X; L_\bullet), \tag{2.11}$$

and the surgery exact sequence (2.1) is isomorphic to the left part of the algebraic surgery exact sequence (2.9) (see [8, Proposition 7.1.4]).

For the case of a compact topological manifold X with boundary ∂X the algebraic surgery exact sequences for the relative case and for the *rel* ∂ case are contained in the following commutative diagram (see [7] and [8, §7])

$$\begin{array}{ccccccccc}
 & \vdots & & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots & L_{n+1}(\pi) & \to & \mathcal{S}^{\partial}_{n+1}(X, \partial X) & \to & H_n(X; \mathbf{L}_\bullet) & \to & L_n(\pi) & \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots & L^{rel}_{n+1} & \to & \mathcal{S}_{n+1}(X, \partial X) & \to & H_n(X, \partial X; \mathbf{L}_\bullet) & \to & L^{rel}_n & \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots & L_n(\rho) & \to & \mathcal{S}_n(\partial X) & \to & H_{n-1}(\partial X; \mathbf{L}_\bullet) & \to & L_{n-1}(\rho) & \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & \vdots & & \vdots & & \vdots & & \vdots &
\end{array} \tag{2.12}$$

where $\pi = \pi_1(X)$, $\rho = \pi_1(\partial X)$, and $L^{rel}_* = L_*(\rho \to \pi)$. Diagram (2.12) is realized on the spectrum level (see [8], [1], and [3]). We denote by $\mathbb{S}(X, \partial X)$ the homotopical cofiber of the map

$$(X/\partial X)_+ \wedge \mathbf{L}_\bullet \to \mathbb{L}(\rho \to \pi),$$

and by $\mathbb{S}^{\partial}(X, \partial X)$ the homotopical cofiber of the map

$$X_+ \wedge \mathbf{L}_\bullet \to \mathbb{L}(\pi).$$

We have

$$\pi_i(\mathbb{S}^{\partial}(X, \partial X)) = \mathcal{S}_i^{\partial}(X, \partial X)$$

and

$$\pi_i(\mathbb{S}(X, \partial X)) = \mathcal{S}_i(X, \partial X).$$

For a topological manifold X the left part of the upper row of diagram (2.12) is isomorphic to the exact sequence (2.2). The left part of middle row of diagram (2.12) is isomorphic to exact sequence (2.3).

In particular, we have the isomorphisms

$$S^{TOP}(X, \partial X) \cong S_{n+1}(X, \partial X), \quad S_{\partial}^{TOP}(X, \partial X) \cong S_{n+1}^{\partial}(X, \partial X), \tag{2.13}$$

and

$$\mathcal{T}^{TOP}(X, \partial X) \cong H_n(X, \partial X; \mathbf{L}_\bullet), \quad \mathcal{T}_{\partial}^{TOP}(X, \partial X) \cong H_n(X; \mathbf{L}_\bullet). \tag{2.14}$$

Consider the composition

$$L_{n+1}(\pi_1(X)) \to L_{n+1}(\pi_1(\partial X) \to \pi_1(X)) \to \mathcal{S}^{TOP}(X, \partial X) \tag{2.15}$$

where the first map lies in the relative exact sequence of L-groups for the map $\pi_1(\partial X) \to \pi_1(X)$ and the second map lies in (2.3). It follows from (2.12) that the composition (2.15) is realized by a map of spectra (see also [3])

$$\mathbb{L}(\pi_1(X)) \to \mathbb{S}(X, \partial X) \tag{2.16}$$

and the cofiber of the map in (2.16) is denoted by $\mathbb{TS}(X, \partial X)$. We shall denote

$$\pi_n(\mathbb{TS}(X, \partial X)) = \mathcal{TS}_n(X, \partial X)$$

and we get an isomorphism [3]

$$\mathcal{TS}_{n+1}(X, \partial X) \cong \mathcal{TS}^{TOP}(X, \partial X).$$

Note that in a similar way (see [3]) it is possible to describe the spectrum $\mathbb{TS}(X, \partial X)$ as the homotopical cofiber of any of the following maps

$$\mathbb{S}(\partial X) \to \Sigma(X_+ \wedge \mathbf{L}_\bullet) \text{ and } (X/\partial X)_+ \wedge \mathbf{L}_\bullet) \to \Sigma\mathbb{L}(\pi_1(\partial X)). \tag{2.17}$$

3. Splitting problem for a manifold with boundary.

Let (X, Y, ξ) be a codimension $q(=1,2)$ manifold pair in the sense of Ranicki (see [8, page 570]), i.e. a locally flat closed submanifold Y is given with a normal block bundle

$$\xi = \xi_{Y \subset X} : Y \to \widetilde{BTOP}(q)$$

for which we have a decomposition of the closed manifold

$$X = D(\xi) \cup_{S(\xi)} \overline{X \setminus D(\xi)}.$$

where $D(\xi)$ is the total space of the normal block bundle with the boundary $S(\xi)$. In accordance with [8, p. 570] the pair (X, Y) has an underlying structure of an $(n, n-q)$-dimensional t-normal geometric Poincaré pair with the associated (D^q, S^{q-1}) fibration

$$(D^q, S^{q-1}) \to (D(\xi), S(\xi)) \to Y. \tag{3.1}$$

The fibration (3.1) provides transfer maps on the spectrum level (see [1], [6], [11], and [12])

$$p^\sharp : \mathbb{L}(\pi_1(Y)) \to \Omega^q \mathbb{L}\left(\pi_1(S(\xi)) \to \pi_1(D(\xi))\right) \tag{3.2}$$

and

$$p_1^\sharp : \mathbb{L}(\pi_1(Y)) \to \Omega^{q-1} \mathbb{L}\left(\pi_1(S(\xi))\right). \tag{3.3}$$

Transfer maps (3.2) and (3.3) fit in a homotopy commutative diagram of spectra

(3.4)

$$\begin{array}{ccccc} \mathbb{L}(\pi_1(Y)) & \xrightarrow{p^\sharp} & \Omega^q\mathbb{L}(\pi_1(S(\xi)) \to \pi_1(D(\xi))) & \to & \Omega^q\mathbb{L}(\pi_1(X \setminus Y) \to \pi_1(X)) \\ & p_1^\sharp \searrow & \downarrow & & \downarrow \\ & & \Omega^{q-1}\mathbb{L}(\pi_1(S(\xi))) & \to & \Omega^{q-1}\mathbb{L}(\pi_1(X \setminus Y)), \end{array}$$

where the horizontal maps of the right square are induced by the horizontal maps of F and the vertical maps are maps from cofibrations of spectra as in (2.8) for the vertical maps of the square F.

The spectrum $\mathbb{LS}(F)$ for splitting obstruction groups of the manifold pair $Y \subset X$ and the spectrum $\mathbb{LP}(F)$ for surgery obstruction groups of the manifold pair fit in the homotopy commutative diagram of spectra

$$\begin{array}{ccccc} \Omega\mathbb{L}(\pi_1(Y)) & \longrightarrow & \Omega^{q+1}\mathbb{L}(\pi_1(X \setminus Y) \to \pi_1(X)) & \longrightarrow & \mathbb{LS}(F) \\ {\scriptstyle =}\downarrow & & \downarrow & & \downarrow \\ \Omega\mathbb{L}(\pi_1(Y)) & \longrightarrow & \Omega^q\mathbb{L}(\pi_1(X \setminus Y)) & \longrightarrow & \mathbb{LP}(F) \end{array} \tag{3.5}$$

where the left horizontal maps are compositions from diagram (3.4) and the right square is the pullback (see [1], [10], and [11]). In particular, we have the isomorphisms

$$\pi_n(\mathbb{LS}(F)) \cong LS_n(F), \ \pi_n(\mathbb{LP}(F)) \cong LP_n(F).$$

A topological normal map [8, §7.2]

$$((f,b),(g,c)) : (M,N) \to (X,Y)$$

to the manifold pair (X,Y,ξ) is represented by a normal map (f,b) to the manifold X which is transversal to Y with $N = f^{-1}(Y)$, and (M,N) is a topological manifold pair with the normal block bundle

$$\nu : N \xrightarrow{f|_N} Y \xrightarrow{\xi} \widetilde{BTOP}(q).$$

Additionally, the following conditions are satisfied:

(i) the restriction

$$(f,b)|_N = (g,c) : N \to Y$$

is a normal map;

(ii) the restriction

$$(f,b)|_P = (h,d) : (P,S(\nu)) \to (Z,S(\xi))$$

is a normal map to the pair $(Z,S(\xi))$, where

$$P = \overline{M \setminus D(\nu)}, \quad Z = \overline{X \setminus D(\xi)};$$

(iii) the restriction

$$(h,d)|_{S(\nu)} : S(\nu) \to S(\xi)$$

coincides with the induced map

$$(g,c)^! : S(\nu) \to S(\xi),$$

and $(f,b) = (g,c)^! \cup (h,d)$.

The normal maps to (X,Y,ξ) are called t-triangulations of the manifold pair (X,Y) and the set of concordance classes of t-triangulations of the pair (X,Y,ξ) coincides with the set of t-triangulations of the manifold X [8, Proposition 7.2.3].

An s-triangulation of a manifold pair (X,Y,ξ) in topological category [8, p. 571] is a t-triangulation of this pair for which the maps

$$f : M \to X, \ g : N \to Y, \text{ and } (P,S(\nu)) \to (Z,S(\xi)) \tag{3.6}$$

are simple homotopy equivalences (s-triangulations). The set of concordance classes of s-triangulations is denoted by $\mathcal{S}^{TOP}(X,Y,\xi)$ (see [8, page 571]). Natural forgetful maps

$$\mathcal{S}^{TOP}(X,Y,\xi) \to \mathcal{S}^{TOP}(X) \text{ and } \mathcal{S}^{TOP}(X,Y,\xi) \to \mathcal{T}^{TOP}(X) \tag{3.7}$$

are well-defined (see [8, §7.2]). We have also the maps of taking obstruction (see [8, page 572])

$$\mathcal{S}^{TOP}(X) \to LS_{n-q}(F) \text{ and } \mathcal{T}^{TOP}(X) \to LP_{n-q}(F). \tag{3.8}$$

The maps in (3.7) and (3.8) are realized on the level of spectra (see [1], [8 §7.2], and [11]). We shall denote by $\mathbb{S}(X, Y, \xi)$ the homotopy cofiber of the map

$$X_+ \wedge \mathbf{L}_\bullet \to \Sigma^q \mathbb{LP}(F)$$

and by $\mathcal{S}_i(X, Y, \xi) = \pi_i(\mathbb{S}(X, Y, \xi))$ its homotopy groups. We have an isomorphism

$$\mathcal{S}_{n+1}(X, Y, \xi) \cong S^{TOP}(X, Y, \xi). \tag{3.9}$$

The maps from (3.7) and (3.8) fit in several diagrams of exact sequences which are given in [8, Proposition 7.2.6]. The diagram

$$\begin{array}{ccccccccccccc}
\to & L_{n+1}(\pi_1(X)) & & \to & & LS_{n-q}(F) & & \to & & \mathcal{S}_n(X,Y,\xi) & & \to \\
& \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
& & & \mathcal{S}_{n+1}(X) & & & & LP_{n-q}(F) & & & & & , \\
& \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\to & \mathcal{S}_{n+1}(X,Y,\xi) & & \longrightarrow & & H_n(X;\mathbf{L}_\bullet) & & \longrightarrow & & L_n(\pi_1(X)) & & \to
\end{array} \tag{3.10}$$

from [8, Proposition 7.2.6] is realized on the spectrum level with the left part ($i \geq n$) which is isomorphic to a geometrically defined diagram (see [8, page 582]) containing structure sets $\mathcal{S}^{TOP}(X), \mathcal{S}^{TOP}(X, Y, \xi), \mathcal{T}^{TOP}(X)$, in accordance with isomorphisms (2.11) and (3.9). Note here that the geometric version of diagram (3.10) also contains the maps from (3.7) and (3.8).

Let

$$(Y, \partial Y) \subset (X, \partial X) \tag{3.11}$$

be a codimension q manifold pair with boundary. A manifold pair (3.11) with boundaries defines a pair of closed manifolds $\partial Y \subset \partial X$ with a pushout square

$$F_\partial = \begin{pmatrix} \pi_1(S(\partial\xi)) & \longrightarrow & \pi_1(\partial X \setminus \partial Y) \\ \downarrow & & \downarrow \\ \pi_1(\partial Y) & \longrightarrow & \pi_1(\partial X) \end{pmatrix} \tag{3.12}$$

of fundamental groups for the splitting problem. A natural inclusion $\delta : \partial X \to X$ induces a map of $\Delta : F_\partial \to F$ of squares of fundamental groups.

In the *rel* ∂-case we consider t-triangulations

$$(f, \partial f) : (M, \partial M) \to (X, \partial X) \tag{3.13}$$

which are split on the boundary along ∂Y. The classes of concordance relative to the boundary of such maps give the set $\mathcal{T}^{TOP}_\partial(X, \partial X)$ (see [8, §7.2]) and the map

$$\mathcal{T}^{TOP}_\partial(X, \partial X) \to LP_{n-q}(F) \tag{3.14}$$

defines a *rel* ∂ codimension q splitting obstruction along $Y \subset X$ (see [8, §7.2]).

In a similar way (see [8, §7.2]) we can consider an s-triangulation of pairs (3.12) which is split along the boundary. The set of concordance *rel* ∂ classes is $\mathcal{S}^{TOP}_\partial(X, \partial X)$ and a *rel* ∂ codimension q splitting obstruction gives a map

$$\mathcal{S}^{TOP}_\partial(X, \partial X) \to LS_{n-q}(F). \tag{3.15}$$

As in the case of closed manifolds denote by $\mathcal{S}_\partial^{TOP}(X, Y, \xi)$ the set of classes of concordance *rel* ∂ maps which are split along $Y \subset X$.

The algebraic version of surgery exact sequence (2.2) and algebraic versions of the maps (3.14) and (3.15) fit in the commutative braid of exact sequences

(3.16)

$$
\begin{array}{ccccccccccc}
\to & L_{n+1}(\pi_1(X)) & & \to & & LS_{n-q}(F) & & \to & & \mathcal{S}_n^\partial(X,Y,\xi) & \to \\
& \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\
& & & \mathcal{S}_{n+1}^\partial(X,\partial X) & & & & LP_{n-q}(F) & & & \\
& \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\
\to & \mathcal{S}_{n+1}^\partial(X,Y,\xi) & & \longrightarrow & & H_n(X;\mathbf{L}_\bullet) & & \longrightarrow & & L_n(\pi_1(X)) & \to
\end{array}
$$

Diagram (3.16) is realized on the level of spectra and for $\partial X = \emptyset$ coincides with the diagram (3.10). The left part ($i \geq n$) of diagram (3.16) is isomorphic to geometrically defined diagram similarly to diagram (3.10). In particular,

$$
\mathcal{S}_i^\partial(X,Y,\xi) = \pi_i(\mathbb{S}^\partial(X,Y,\xi)), \text{and } \mathcal{S}_{n+1}^\partial(X,Y,\xi) \cong \mathcal{S}_\partial^{TOP}(X,Y,\xi). \tag{3.17}
$$

Denote by $LS_*(\Delta) = LS_*(F_\partial \to F)$ and $LP_*(\Delta) = LP_*(F_\partial \to F)$ the relative groups for the map of squares $\Delta : F_\partial \to F$ which is induced by the natural inclusion $\delta : \partial X \to X$. It follows from functoriality of diagram (3.5) that these relative groups are realized on the level of spectra. We have cofibrations of Ω-spectra

$$
\mathbb{LS}(F_\partial) \to \mathbb{LS}(F) \to \mathbb{LS}(\Delta) \tag{3.18}
$$

and

$$
\mathbb{LP}(F_\partial) \to \mathbb{LP}(F) \to \mathbb{LP}(\Delta) \tag{3.19}
$$

where

$$
\pi_n(\mathbb{LS}(\Delta)) \cong LS_n(F_\partial \to F) \text{ and } \pi_n(\mathbb{LP}(\Delta)) \cong LP_n(F_\partial \to F).
$$

These groups fit in the commutative diagram of exact sequences

(3.20)

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots L_{n+1}(\rho) & \to & LS_{n-q}(F_\partial) & \to & LP_{n-q}(F_\partial) & \to & L_n(\rho) & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots L_{n+1}(\pi) & \to & LS_{n-q}(F) & \to & LP_{n-q}(F) & \to & L_n(\pi) & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots L_{n+1}(\rho \to \pi) & \to & LS_{n-q}(\Delta) & \to & LP_{n-q}(\Delta) & \to & L_n(\rho \to \pi) & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\vdots & & \vdots & & \vdots & & \vdots &
\end{array}
$$

where $\pi = \pi_1(X)$ and $\rho = \pi_1(\partial X)$. Diagram (3.20) is realized on the level of spectra and the two middle columns are homotopy long exact sequences of cofibrations (3.18) and (3.19).

Now consider relative structure groups for a codimension q manifold pair with boundaries (3.11). We have a normal block bundle $(\xi, \partial\xi)$ over the pair $(Y, \partial Y)$ and a decomposition

$$
(X, \partial X) = (D(\xi) \cup_{S(\xi)} Z, D(\partial\xi) \cup_{S(\partial\xi)} \partial_+ Z) \tag{3.21}
$$

where $(Z; \partial_+ Z, S(\xi); S(\partial\xi))$ is a manifold triad. Note here that $\partial_+ Z = \overline{\partial X \setminus D(\partial\xi)}$.

A topological normal map (3.13) of manifold pairs with boundaries provides a normal block bundle $(\nu, \partial\nu)$ over the pair $(N, \partial N)$, where (see [8, p. 570])

$$(N, \partial N) = (f^{-1}(Y), (\partial f)^{-1}(\partial Y)).$$

We have the following decomposition

$$(M, \partial M) = (D(\nu) \cup_{S(\nu)} P, D(\partial\nu) \cup_{S(\partial\nu)} \partial_+ P) \tag{3.22}$$

where $(P; \partial_+ P, S(\nu); S(\partial\nu))$ is a manifold triad.

Let

$$(f, \partial f) : (M, \partial M) \to (X, \partial X)$$

be a normal map of a codimension q pair with boundary $(N, \partial N) \subset (M, \partial M)$ to a codimension q pair $(Y, \partial Y) \subset (X, \partial X)$. It is an s-triangulation if the maps $f : M \to X$ and $\partial f : \partial M \to \partial X$ are s-triangulations of corresponding codimension q pairs. We shall denote the set of concordance classes of s-triangulations of the codimension q manifold pair (3.11) by

$$\mathcal{S}^{TOP}(X, Y; \partial) = \mathcal{S}^{TOP}(X, \partial X; Y, \partial Y; \xi, \partial(\xi)).$$

The relative surgery theory (see [8, §7.2], [10, §11], and [11]) guarantees that this structure set fits in the following exact sequences.

$$\cdots \to \mathcal{S}^{TOP}(X, Y; \partial) \to \mathcal{T}^{TOP}(X, \partial X) \to LP_{n-q}(\Delta) \tag{3.23}$$

and

$$\cdots \to \mathcal{S}^{TOP}(X, Y; \partial) \to \mathcal{S}^{TOP}(X, \partial X) \to LS_{n-q}(\Delta) \tag{3.24}$$

Proposition 1. *There exists an Ω-spectrum $\mathbb{S}(X, Y; \partial)$ with homotopy groups*

$$\mathcal{S}_i(X, Y; \partial) \cong \pi_i(\mathbb{S}(X, Y; \partial)) \text{ and } \mathcal{S}_{n+1}(X, Y; \partial) \cong \mathcal{S}^{TOP}(X, Y; \partial)). \tag{3.25}$$

There are algebraic versions of exact sequences (3.23) and (3.24)

$$\cdots \to \mathcal{S}_{n+1}(X, Y; \partial) \to H_n(X, \partial X; \mathbf{L}_\bullet) \xrightarrow{\lambda} LP_{n-q}(\Delta) \to \cdots \tag{3.26}$$

and

$$\cdots \to \mathcal{S}_{n+1}(X, Y; \partial) \to \mathcal{S}_{n+1}(X, \partial X) \to LS_{n-q}(\Delta) \to \cdots \tag{3.27}$$

which are realized on the spectrum level by cofibrations

$$(X/\partial X)_+ \wedge \mathbf{L}_\bullet \to \Sigma^q \mathbb{LP}(\Delta) \to \mathbb{S}(X, Y; \partial) \tag{3.29}$$

and

$$\mathbb{S}(X, \partial X) \to \Sigma^{q+1} \mathbb{LS}(\Delta) \to \Sigma S(X, Y; \partial), \tag{3.30}$$

respectively.

Proof. Commutative diagram (2.12) is generated by a homotopy commutative diagram of spectra
(3.31)

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & (\partial X)_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \mathbb{L}(\rho) & \longrightarrow & \mathbb{S}(\partial X) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & X_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \mathbb{L}(\pi) & \longrightarrow & \mathbb{S}^\partial(X, \partial X) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & (X/\partial X)_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \mathbb{L}(\rho \to \pi) & \longrightarrow & \mathbb{S}(X, \partial X) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

in which each row and column is a cofibration sequence. Commutative diagram (3.20) is generated by a homotopy commutative diagram of spectra

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathbb{LS}(F_\partial) & \longrightarrow & \mathbb{LP}(F_\partial) & \longrightarrow & \Sigma^{-q}\mathbb{L}(\rho) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathbb{LS}(F) & \longrightarrow & \mathbb{LP}(F) & \longrightarrow & \Sigma^{-q}\mathbb{L}(\pi) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathbb{LS}(\Delta) & \longrightarrow & \mathbb{LP}(\Delta) & \longrightarrow & \Sigma^{-q}\mathbb{L}(\rho \to \pi) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array} \tag{3.32}$$

in which each row and column is a cofibration sequence. Consider a homotopy commutative square of spectra

$$\begin{array}{ccc} (\partial X)_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \Sigma^q \mathbb{LP}(F_\partial) \\ \downarrow & & \downarrow \\ X_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \Sigma^q \mathbb{LP}(F) \end{array} \tag{3.33}$$

in which the vertical maps are induced by the inclusion δ and the horizontal maps follows from diagram (3.10) of the manifold pair $\partial Y \subset \partial X$ and from diagram (3.16),

respectively. Denote by $\mathbb{S}(X,Y;\partial)$ a spectrum fitting in the diagram extending the square (3.33) by cofibration sequences
(3.34)

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & (\partial X)_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \Sigma^q \mathbb{LP}(F_\partial) & \longrightarrow & \mathbb{S}(\partial X, \partial Y, \partial \xi) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & X_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \Sigma^q \mathbb{LP}(F) & \longrightarrow & \mathbb{S}^\partial(X,Y,\xi) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & (X/\partial X)_+ \wedge \mathbf{L}_\bullet & \longrightarrow & \Sigma^q \mathbb{LP}(\Delta) & \longrightarrow & \mathbb{S}(X,Y;\partial) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Let $\mathcal{S}_i(X,Y;\partial) = \pi_i(\mathbb{S}(X,Y;\partial))$. The papers [7], [11, §7.2], and [14, §7.A] provide commutative squares

$$\begin{array}{cc}
H_{n+k}(X;\mathbf{L}_\bullet) & \to LP_{n-q+k}(\Delta) \\
t\downarrow\cong & \downarrow= \\
\mathcal{T}^{TOP}(X \times D^k, \partial(X \times D^k)) & \to LP_{n-q+k}(\Delta)
\end{array}$$

and

$$\begin{array}{cc}
H_n(X,\partial X;\mathbf{L}_\bullet) & \to LP_{n-q}(\Delta) \\
t\downarrow\cong & \downarrow= \\
\mathcal{T}^{TOP}(X,\partial X) & \to LP_{n-q}(\Delta).
\end{array}$$

Note that the exact sequence (3.26) is obtained by applying the functor π_0 to the bottom cofibration in (3.34). Now, using geometric description of surgery spectra (see [7] and [14]), an element $x \in \mathcal{S}_{n+1}(X,Y,\partial)$ is defined by a pair (y,z), consisting of a normal map bordism class $y \in H_n(X,\partial X;\mathbf{L}_\bullet)$, for which $\lambda(y) = 0 \in LP_{n-q}(\Delta)$, and a particular solution z of the associated surgery problem for manifold pairs with boundaries that defines a class of equivalence

$$\{(f : M \to X)\} \in \mathcal{S}^{TOP}(X,Y,\partial).$$

We obtain the map

$$\sigma : \mathcal{S}_{n+1}(X,Y,\partial) \to \mathcal{S}^{TOP}(X,Y,\partial).$$

Recall, that in geometrically defined exact sequence (3.23) the map $LP_{n-q+1}(\Delta) \to \mathcal{S}^{TOP}(X,Y,\partial)$ is an action. Now from the definition of the map σ follows the commutative diagram (see [7] and [10])

$$\begin{array}{ccccc}
LP_{n-q+1}(\Delta) & \to & \mathcal{S}_{n+1}(X,Y,\partial) & \to & H_n(X,\partial X;\mathbf{L}_\bullet) \\
\downarrow= & & \sigma\downarrow & & t\downarrow\cong \\
LP_{n-q+1}(\Delta) & \to & \mathcal{S}^{TOP}(X,Y,\partial) & \to & \mathcal{T}^{TOP}(X,\partial X).
\end{array}$$

Using the Five Lemma we obtain an isomorphism between (3.23) and (3.26). The case of exact sequence (3.27) follows from a homotopy commutative diagram of cofibrations

(3.35)

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & \mathbb{S}(\partial X, \partial Y, \partial\xi) \arrow[r] \arrow[d] & \mathbb{S}(\partial X) \arrow[r] \arrow[d] & \Sigma^{q+1}\mathbb{LS}(F_\partial) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathbb{S}^\partial(X, Y, \xi) \arrow[r] \arrow[d] & \mathbb{S}^\partial(X, \partial X) \arrow[r] \arrow[d] & \Sigma^{q+1}\mathbb{LS}(F) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathbb{S}(X, Y; \partial) \arrow[r] \arrow[d] & \mathbb{S}(X, \partial X) \arrow[r] \arrow[d] & \Sigma^{q+1}\mathbb{LS}(\Delta) \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

which is similar to (3.34). Diagram (3.35) follows from consideration of the cofibration sequences of the right upper square in (3.35). □

Consider the composition

$$LS_*(F_\partial) \to LP_*(F_\partial) \to LP_*(F) \tag{3.36}$$

of geometrically defined maps from diagram (3.20). The composition (3.36) is realized by a map of spectra

$$\mathbb{LS}(F_\partial) \to \mathbb{LP}(F) \tag{3.37}$$

which is the composition of maps from diagram (3.32). We denote the homotopical cofiber of the map (3.37) by $\mathbb{LPS}(\Delta)$ and its homotopy groups by

$$LPS_i(\Delta) = \pi_i(\mathbb{LPS}(\Delta)). \tag{3.38}$$

In particular, we have a cofibration

$$\mathbb{LS}(F_\partial) \to \mathbb{LP}(F) \to \mathbb{LPS}(\Delta). \tag{3.39}$$

Theorem 2. *There exists the following cofibration of spectra*

$$\mathbb{S}(X, Y, \partial) \to \mathbb{TS}(X, \partial X) \to \Sigma^{q+1}\mathbb{LPS}(\Delta). \tag{3.40}$$

Proof. Consider the following homotopy commutative diagram of spectra

$$\begin{tikzcd}
\mathbb{S}(\partial X, \partial Y, \partial\xi) \arrow[r] \arrow[d] & \mathbb{S}^\partial(X, Y, \xi) \arrow[r] \arrow[d] & \mathbb{S}(X, Y; \partial) \arrow[d] \\
\mathbb{S}(\partial X) \arrow[r] \arrow[d] & \Sigma(X_+ \wedge \mathbf{L}_\bullet) \arrow[r] \arrow[d] & \mathbb{TS}(X, \partial X) \arrow[d] \\
\Sigma^{q+1}\mathbb{LS}(F_\partial) \arrow[r] & \Sigma^{q+1}\mathbb{LP}(F) \arrow[r] & \Sigma^{q+1}\mathbb{LPS}(\Delta)
\end{tikzcd} \tag{3.41}$$

in which all rows and columns are cofibration sequences. The left bottom square of (3.41) follows from the commutative diagram

$$
\begin{array}{ccccc}
\mathbb{S}(\partial X, \partial Y, \partial\xi) & \xrightarrow{\simeq} & \mathbb{S}(\partial X, \partial Y, \partial\xi) & \longrightarrow & \mathbb{S}^{\partial}(X, Y, \xi) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{S}(\partial X) & \longrightarrow & \Sigma((\partial X)_+ \wedge \mathbf{L}_\bullet) & \longrightarrow & \Sigma(X_+ \wedge \mathbf{L}_\bullet) \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma^{q+1}\mathbb{L}\mathbb{S}(F_\partial) & \longrightarrow & \Sigma^{q+1}\mathbb{L}\mathbb{P}(F_\partial) & \longrightarrow & \Sigma^{q+1}\mathbb{L}\mathbb{P}(F)
\end{array} \tag{3.42}
$$

in which the commutative part consisting of the two left squares follows from the diagram (3.10) on the spectrum level for the manifold pair $\partial Y \subset \partial X$ and the commutative part consisting of the two right squares fits in (3.34). The vertical columns of (3.42) are cofibrations. The left column of (3.41) coincides with the left column of (3.42), and the middle column of (3.41) coincides with the right column of (3.42). The right vertical maps in (3.41) are defined as map of cofibers of horizontal maps in accordance with (2.17), (3.35), and (3.39). The left upper horizontal map in (3.41) is the composition

$$
\mathbb{S}(\partial X, \partial Y, \partial\xi) \xrightarrow{\simeq} \mathbb{S}(\partial X, \partial Y, \partial\xi) \to \mathbb{S}^{\partial}(X, Y, \xi) \tag{3.43}
$$

from the diagram (3.42). The right column of cofibration (3.41) is cofibration (3.40). □

Corollary 3. *There exists the following long exact sequence*

$$
\cdots \to \mathcal{S}_n(X, Y; \partial) \to \mathcal{TS}_n(X, \partial X) \to LPS_{n-q-1}(\Delta) \to \cdots \tag{3.44}
$$

Proof. The exact sequence (3.44) is the homotopy long exact sequence of the cofibration (3.40). □

Corollary 4. *There is a map*

$$
\Theta : \mathcal{TS}^{TOP}(X, \partial X) \to LPS_{n-q}(\Delta)
$$

of obstructions to splitting along the submanifold with boundary $(Y, \partial Y)$. For a representative $z = ((f, b), (\partial f, \partial b))$ we have $\Theta(z) = 0$ if and only if the class of z in $\mathcal{TS}^{TOP}(X, \partial X)$ contains a representative which is split along $(Y, \partial Y) \subset (X, \partial X)$.

Proof. We have the commutative diagram

$$
\begin{array}{ccc}
\mathcal{S}_{n+1}(X, Y, \partial) & \longrightarrow & \mathcal{S}^{TOP}(X, Y; \partial) \\
\downarrow & & \downarrow \\
\mathcal{TS}_{n+1}(X, \partial X) & \longrightarrow & \mathcal{TS}^{TOP}(X, \partial X)
\end{array} \tag{3.45}
$$

in which the horizontal maps are isomorphisms, the right vertical map is a natural forgetful map, and the left vertical map follows from (3.41). Now the diagram (3.41) and the exact sequence (3.44) provide the result of the Corollary. □

From now on we describe algebraic properties of LPS_*-groups and their relations to splitting and surgery obstruction groups.

Theorem 5. *There exists a braid of exact sequences*

$$
\begin{array}{ccccccccccccc}
\to & & LP_{n+1}(\Delta) & & \to & & L_{n+q}(\rho) & & \to & & LS_{n-1}(F_\partial) & & \to \\
& \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
& & & & LP_n(F_\partial) & & & & LPS_n(\Delta) & & & & , \\
& \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\to & & LS_n(F_\partial) & & \longrightarrow & & LP_n(F) & & \longrightarrow & & LP_n(\Delta) & & \to
\end{array}
\tag{3.46}
$$

$$
\begin{array}{ccccccccccccc}
\to & & L_{n+q+1}(\pi) & & \to & & LS_n(\Delta) & & \to & & LP_n(\Delta) & & \to \\
& \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
& & & & L_{n+q+1}(\rho \to \pi) & & & & LPS_n(\Delta) & & & & , \\
& \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\to & & LP_{n+1}(\Delta) & & \longrightarrow & & L_{n+q}(\rho) & & \longrightarrow & & L_{n+q}(\pi) & & \to
\end{array}
\tag{3.47}
$$

and

$$
\begin{array}{ccccccccccccc}
\to & & LS_n(F_\partial) & & \to & & LP_n(F) & & \to & & L_{n+q}(\pi) & & \to \\
& \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
& & & & LS_n(F) & & & & LPS_n(\Delta) & & & & , \\
& \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\to & & L_{n+q+1}(\pi) & & \longrightarrow & & LS_n(\Delta) & & \longrightarrow & & LS_{n-1}(F_\partial) & & \to
\end{array}
\tag{3.48}
$$

where $\rho = \pi_1(\partial X)$ and $\pi = \pi_1(X)$. Diagrams (3.46) – (3.48) are realized on the spectrum level.

Proof. It follows from the definition of spectra $\mathbb{LPS}$ and diagram (3.32) that we have a homotopy commutative diagram of spectra

$$
\begin{array}{ccccc}
\mathbb{LS}(F_\partial) & \to & \mathbb{LP}(F_\partial) & \to & \Sigma^{-q}\mathbb{L}(\rho) \\
\downarrow = & & \downarrow & & \downarrow \\
\mathbb{LS}(F_\partial) & \to & \mathbb{LP}(F) & \to & \mathbb{LPS}(\Delta)
\end{array}
\tag{3.49}
$$

where the rows are cofibrations and the right vertical map is induced by the two left vertical maps (see [9]). The right square in (3.49) is a pullback square since fibers of horizontal maps are naturally homotopy equivalent. Hence the right square in (3.49) is a pushout square and homotopy long exact sequences of this square give the braid of exact sequences (3.46). From diagram (3.32) and [5, Lemma 2] we conclude that the spectrum $\mathbb{LPS}(\Delta)$ fits in the cofibrations of spectra

$$\Sigma^{-1}\mathbb{LP}(\Delta) \to \Sigma^{-q}\mathbb{L}(\rho) \to \mathbb{LPS}(\Delta)$$

and

$$\Sigma^{-q-1}\mathbb{L}(\pi) \to \mathbb{LS}(\Delta) \to \mathbb{LPS}(\Delta).$$

Now the same line of arguments as for the braid of exact sequences (3.46) provides diagrams (3.47) and (3.48). $\square$

Corollary 6. *The groups $LPD_*(\Delta)$ fit in the following exact sequences*

$$\cdots \to LS_n(F_\partial) \to LP_n(F) \to LPS_n(\Delta) \to \cdots, \tag{3.50}$$

$$\cdots \to LP_{n+1}(\Delta) \to L_{n+q}(\rho) \to LPS_n(\Delta) \to \cdots, \tag{3.51}$$

and

$$\cdots \to L_{n+q+1}(\pi) \to LS_n(\Delta) \to LPS_n(\Delta) \to \cdots, \tag{3.52}$$

which are realized on the level of spectra.

Proof. These sequences fit in the diagrams of Theorem 6. □

Corollary 7. *Let $\Delta : F_\partial \to F$ be an isomorphism of pushout squares. Then we have isomorphisms*

$$LPS_n(\Delta) \cong L_{n+q}(\rho) \cong L_{n+q}(\pi).$$

Proof. The result follows immediately from the exact sequences of Corollary 6. □

The next theorem describes relations between the obstruction groups LPS_* and different structure sets which arise naturally for the manifold pair with boundaries.

Theorem 8. *There exist the following braids of exact sequences*

(3.53)

$$\begin{array}{ccccccccccc}
\to & \mathcal{S}_n(X,Y;\partial) & & \to & & \mathcal{TS}_n(X,\partial X) & & \to & & L_{n-1}(\pi) & \to \\
 & \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & \searrow & \\
 & & & \mathcal{S}_n(X,\partial X) & & & & LPS_{n-q-1}(\Delta) & & & , \\
 & \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & \nearrow & \\
\to & L_n(\pi) & & \longrightarrow & & LS_{n-q-1}(\Delta) & & \longrightarrow & & \mathcal{S}_{n-1}(X,Y;\partial) & \to
\end{array}$$

and

(3.54)

$$\begin{array}{ccccccccccc}
\to & \mathcal{S}_{n+1}(X,Y;\partial) & & \to & & H_n(X,\partial X;\mathbf{L}_\bullet) & & \to & & L_{n-1}(\rho) & \to \\
 & \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & \searrow & \\
 & & & \mathcal{TS}_{n+1}(X,\partial X) & & & & LP_{n-q}(\Delta) & & & , \\
 & \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & \nearrow & \\
\to & L_n(\rho) & & \longrightarrow & & LPS_{n-q}(\Delta) & & \longrightarrow & & \mathcal{S}_n(X,Y;\partial) & \to
\end{array}$$

where $\rho = \pi_1(\partial X)$ and $\pi = \pi_1(X)$. Diagrams (3.53) and (3.54) are realized on the level of spectra.

Proof. Consider a homotopy commutative diagram of spectra

$$\begin{array}{ccccc}
\mathbb{L}(\pi) & \longrightarrow & \mathbb{S}(X,\partial X) & \longrightarrow & \mathbb{TS}(X,\partial X) \\
{=}\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathbb{L}(\pi) & \longrightarrow & \Sigma^{q+1}\mathbb{LS}(\Delta)) & \longrightarrow & \Sigma^{q+1}\mathbb{LPS}(X,\partial X)
\end{array} \tag{3.55}$$

in which the rows follow from definitions and the columns are obtained from the natural map of the diagram (3.31) to the diagram (3.32). The right square of (3.55) is the pullback and the homotopy long exact sequences of this square give the commutative diagram (3.53). The case of the diagram (3.54) is similar. □

4. Examples.

In this section we compute the LPS_*-groups and natural maps for several geometric examples.

Let $(Y^{n-1}, \partial Y) \subset (X^n, \partial X)$, $n \geq 4$, be a manifold pair with boundaries, where X is a non-trivial I-bundle over the real projective space $\mathbb{R}P^{n-1}$ and the submanifold Y is the restriction of the I-bundle to the projective space $\mathbb{R}P^{n-2} \subset \mathbb{R}P^{n-1}$. The pair $\partial Y \subset \partial X$ coincides with $S^{n-2} \subset S^{n-1}$. We have isomorphisms $\pi_1(X^n) = \mathbb{Z}_2^+$ for n odd and $\pi_1(X^n) = \mathbb{Z}_2^-$ for n even. The group $\mathbb{Z}_2^+$ is a cyclic group of order 2 with the trivial homomorphism of orientation and $\mathbb{Z}_2^-$ is this group with the nontrivial homomorphism of orientation. In the considered case the squares (1.2) and (3.12) are the following squares

$$F^{\pm} = \begin{pmatrix} 1 & \to & 1 \\ \downarrow & & \downarrow \\ \mathbb{Z}_2^{\mp} & \to & \mathbb{Z}_2^{\pm} \end{pmatrix} \tag{4.1}$$

and

$$F_\partial = \begin{pmatrix} 1 \cup 1 & \to & 1 \cup 1 \\ \downarrow & & \downarrow \\ 1 & \to & 1 \end{pmatrix}. \tag{4.2}$$

All horizontal maps in squares (4.1) and (4.2) are isomorphisms.

Theorem 9. *In the considered cases the natural maps*

$$LP_n(F^{\pm}) \to LPS_n(F_\partial \to F^{\pm})$$

fitting in diagrams (3.46) and (3.48) are isomorphisms for $n = 0, 1, 2, 3 \bmod 4$. *Hence we have*

$$LPS_n(F_\partial \to F^+) \cong \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z} \tag{4.3}$$

and

$$LPS_n(F_\partial \to F^-) \cong \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2 \tag{4.4}$$

for $n = 0, 1, 2, 3 \bmod 4$, *respectively.*

Proof. It follows from [10, page 153] that

$$LS_n(F_\partial) = LN_n(1 \cup 1 \to 1) = 0$$

for all n. From this result and the exact sequence (3.50) the first statement of the theorem follows. We have isomorphisms

$$LP_n(F^{\pm}) \cong L_{n+1}(i^!_{\mp})$$

where

$$i_{\mp} : 1 \to \mathbb{Z}_2^{\mp}$$

is the natural inclusion and $L_{n+1}(i^!_{\mp})$ is the relative group of the transfer map (see, for example, [6], [8] and [9]). Now isomorphisms (4.3) and (4.4) follow (see, for example, [9, §3] for the case F^+).

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